merely reversing the direction of the forces in the original system (i.e., changing the sign of the force parameter in the equation of motion). Since the original and the adjoint systems have the same eigenvalues,[1] a reversal of the direction of the forces does not alter the frequencies of vibration of the system. Flutter boundaries of undamped, linearly elastic systems are characterized by the coalescence of two frequencies. Since the frequencies are unaltered, the flutter boundaries are also unaltered due to a reversal of the forces. The previous results may be summed up in two points: 1) anti-adjoint systems never buckle; and 2) frequencies and flutter boundaries of anti-adjoint systems are unaltered by a reversal in the direction of the forces.

Example

A typical example of an anti-adjoint system is given by the equation of motion

$$\mu(\partial^2 w/\partial t^2) + K[w] + P(\partial w/\partial x_j) = 0 \tag{10}$$

and the boundary condition

$$w = 0 \quad \text{on } S \tag{11}$$

This system, along with its application in aeroelasticity, has been discussed by the author in an earlier paper.[2] Where Ref. 2 was based on a two-term Galerkin approximation, the present work is mathematically exact.

Appendix

Let

$$w(x_j,t) = e^{i\omega t}\bar{w}(x_j) \tag{A1}$$

where ω is the frequency of vibration. Substituting Eq. (A1) in Eq. (1)

$$-\mu\omega^2\bar{w} + K[\bar{w}] + PF[\bar{w}] = 0 \tag{A2}$$

Since the original and the adjoint systems have the same frequencies,[1] let

$$u(x_j,t) = e^{i\omega t}\bar{u}(x_j) \tag{A3}$$

Substituting in Eq. (3)

$$-\mu\omega^2\bar{u} + K[\bar{u}] + PF^*[\bar{u}] = 0 \tag{A4}$$

By Green's identity,[3] the original and the adjoint operators are related by

$$\int_V \{-\mu\omega^2\phi + K[\phi] + PF[\phi]\}\psi \mathrm{d}V$$
$$\equiv \int_V \{-\mu\omega^2\psi + K[\psi] + PF^*[\psi]\}\phi \mathrm{d}V \tag{A5}$$

where $\phi(x_j)$ and $\psi(x_j)$ are functions satisfying the original and the adjoint boundary conditions, Eqs. (2) and (4), respectively. Since the boundary conditions are the same for an anti-adjoint system, and $w(x_j,t)$ satisfies these boundary conditions, let

$$\phi(x_j) \equiv w(x_j,t) \qquad \psi(x_j) \equiv w(x_j,t) \tag{A6}$$

Substituting in Eq. (A5) and simplifying

$$\int_V F[w]w\mathrm{d}V = \int_V F^*[w]w\mathrm{d}V \tag{A7}$$

Using Eq. (6) in Eq. (A7)

$$\int_V F[w]w\mathrm{d}V = 0 \tag{A8}$$

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Gas-Particle Flow Past Bodies with Attached Shock Waves

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Introduction

THE problem of supersonic flow of a gas containing small particles (which may consist of water, ice, dust, or metal) past a solid body is of interest because the results obtained from its solution are pertinent to the estimation of the cratering and erosion damage experienced by a flight vehicle when it collides with such particles in the atmosphere. Some recent papers on this subject are those by Probstein and Fassio,[1] Waldman and Reinecke,[2] Spurk and Gerber,[3] Peddieson and Lyu,[4,5] and Lyu and Peddieson.[6] The purpose of the present Note is to put the governing equations for the dispersed phase of a dilute, low-mass-fraction, air-particle suspension (such as the atmosphere) in a form that is convenient for the numerical analysis of flows past symmetric bodies at zero angle of attack having attached shock waves. As an application of the equations, local collection efficiencies are computed for several body shapes and presented graphically.

Governing Equations

Figure 1 depicts the geometry of the problem and serves to define the axial coordinate x, the radial coordinate r, the gas velocity components u and v, the particle-phase velocity components u_p and v_p, the body surface $r_b(x)$, the shock surface $r_s(x)$ the body angle $\theta_b(x)$, the shock angle $\theta_s(x)$, the length of the region in which impacts are to be investigated L, and the freestream velocity U_∞, temperature T_∞, and particle-phase density $\rho_{p\infty}$. If the particle phase is treated as a continuum, body forces and interphase mass transfer are neglected, interphase momentum transfer is assumed to be linear in the difference between the velocity vectors of the two phases, and interphase heat transfer is assumed to be linear in the difference between the temperatures of the two phases (the last two assumptions are made to achieve the maximum simplicity in writing the equations and do not affect any of the subsequent analysis); the governing equations of the dispersed phase for steady axisymmetric flow can be written as

$$(r^j\rho_p u_p)_{,x} + (r^j\rho_p v_p)_{,r} = 0 \tag{1a}$$

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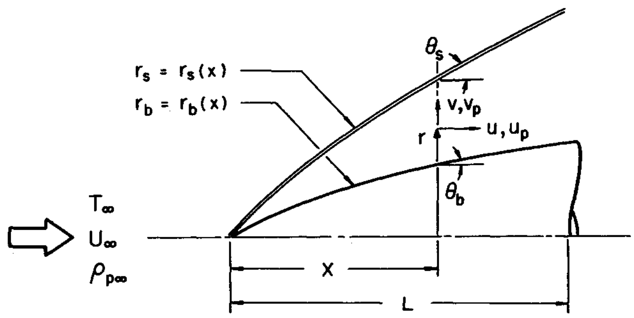

Fig. 1 Geometry and coordinate system.

$$\rho_p(u_p u_{p,x} + v_p u_{p,r}) = \rho_p N_1(u - u_p) \quad (1b)$$

$$\rho_p(u_p v_{p,x} + v_p v_{p,r}) = \rho_p N_1(v - v_p) \quad (1c)$$

$$\rho_p c_p(u_p T_{p,x} + v_p T_{p,r}) = \rho_p c N_2(T - T_p) \quad (1d)$$

where Eq. (1a) results from balance of mass, Eq. (1b) from balance of axial momentum, Eq. (1c) from balance of radial momentum, and Eq. (1d) from balance of energy. In Eqs. (1) ρ_p is the particle-phase density, T and T_p are the temperatures of the gas and particle phases respectively, c is the gas-phase specific heat at constant pressure, c_p is the specific heat of the particle material, N_1 and N_2 are the interphase momentum transfer and heat transfer coefficients respectively, a comma denotes partial differentiation, and $j = 1$. For plane flow the same equations result with $j = 0$. The quantities N_1 and N_2 depend on the microscopic properties of the suspension (see, for example, Marble[7]). They are treated as constants in the present work. The quantities u, v, and T are regarded as known because the behavior of the fluid phase of a dilute, low-mass-fraction suspension is essentially independent of the presence of the particles.

It is convenient to define the following dimensionless variables.

$$\xi = (x/L) \quad (2a)$$

$$\eta = (r/L) \quad (2b)$$

$$F = (u/U_\infty) \quad (2c)$$

$$F_p = (u_p/U_\infty) \quad (2d)$$

$$G = (v/U_\infty) \quad (2e)$$

$$G_p = (v_p/U_\infty) \quad (2f)$$

$$H = (T/T_\infty) \quad (2g)$$

$$H_p = (T_p/T_\infty) \quad (2h)$$

Substituting Eqs. (2) into Eqs. (1) yields

$$(\eta^j Q_p F_p)_{,\xi} + (\eta^j Q_p G_p)_{,\eta} = 0 \quad (3a)$$

$$F_p F_{p,\xi} + G_p F_{p,\eta} = \alpha_1(F - F_p) \quad (3b)$$

$$F_p G_{p,\xi} + G_p F_{p,\eta} = \alpha_1(G - G_p) \quad (3c)$$

$$F_p H_{p,\xi} + G_p H_{p,\eta} = \alpha_2(H - H_p) \quad (3d)$$

where $\alpha_1 = (N_1 L/U_\infty)$ and $\alpha_2 = (c/c_p)(N_2 L/U_\infty)$. The appropriate boundary conditions are

$$F_{ps} = Q_{ps} = H_{ps} = 1 \quad (4a)$$

$$G_{ps} = 0 \quad (4b)$$

Here, and in what follows, a subscript s will denote the value of a variable on the shock surface while a subscript b will denote its value on the body surface.

Equations (3) can be put in a more convenient form by the use of a Von Mises transformation. Toward this end a dimensionless stream function ψ_p satisfying Eq. (3a) is defined such that

$$\eta^j Q_p F_p = -\psi_{p,\eta}, \quad \eta^j Q_p G_p = \psi_{p,\xi} \quad (5)$$

Finding the total differential of ψ_p and reversing the roles of ξ and ψ_p (that is, regarding ξ as a dependent variable and ψ_p as an independent variable) results in

$$d\xi = [d\xi_p/(\eta^j Q_p G_p)] + (F_p d\eta/G_p) \quad (6)$$

from which it follows that

$$\xi_{,\psi_p} = 1/(\eta^j Q_p G_p) \quad (7a)$$

$$\xi_{,\eta} = (F_p/G_p) \quad (7b)$$

Transforming Eqs. (3b-d) and rearranging Eqs. (7) produces

$$G_p F_{p,\eta} = \alpha_1(F - F_p) \quad (8a)$$

$$G_p G_{p,\eta} = \alpha_1(G - G_p) \quad (8b)$$

$$G_p \xi_{,\eta} = F_p \quad (8c)$$

$$G_p H_{p,\eta} = \alpha_2(H - H_p) \quad (8d)$$

$$Q_p = 1/(\eta^j G_p \xi_{,\psi_p}) \quad (8e)$$

Since no derivatives with respect to ψ_p appear in Eqs. (8a-d), they can be treated as ordinary differential equations along the dispersed-phase streamlines (lines of contant ψ_p).

In order to make the radial distance from the body surface to the shock surface constant, it is convenient to define a new independent variable

$$z = (\eta_s - \eta)/(\eta_s - \eta_b) \quad (9)$$

which maps the region $\eta_b \leq \eta \leq \eta_s$ into the region $1 \geq z \geq 0$. It is also useful to replace ψ_p by the coordinate ξ_s at which the particle-phase streamline associated with ψ_p intersects the shock surface. Evaluating Eq. (6) on the shock with the aid of Eqs. (4) yields

$$d\psi_p = -\eta_{so}^j d\eta_{so} = -\eta_{so}^j(d\eta_{so}/d\xi_s) = -\eta_{so}^j \tan(\theta_{so})d\xi_s \quad (10)$$

where $\eta_{so} = \eta_s(\xi_s)$ and $\theta_{so} = \theta_s(\xi_s)$. Transforming Eqs. (8) to the final independent variables ξ_s and z, using Eqs. (9) and (10), one obtains

$$F_{p,z} + (\alpha_1 D_1/D_2)(F - F_p) = 0 \quad (11a)$$

$$G_{p,z} + (\alpha_1 D_1/D_2)\ (G - G_p) = 0 \quad (11b)$$

$$\xi_{,z} + (D_1 F_p/D_2) = 0 \quad (11c)$$

$$H_{p,z} + (\alpha_2 D_1/D_2)\ (H - H_p) = 0 \quad (11d)$$

$$Q_p = [(D_{20}/D_2)\ (D_{30}/D_3)^j/\xi_{,\xi_s}] \quad (11e)$$

where

$$D_1 = \eta_s - \eta_b \quad (12a)$$

$$D_2 = G_p - F_p[(1 - z)\tan(\theta_s) + z\tan(\theta_b)] \quad (12b)$$

$$D_{20} = -\tan(\theta_{so}) \quad (12c)$$

$$D_3 = (1 - z)\eta_s + z\eta_b \quad (12d)$$

$$D_{30} = \eta_{so} \tag{12e}$$

Equations (11) must be solved subject to the boundary conditions

$$F_p(\xi_s, 0) = H_p(\xi_s, 0) = 1 \tag{13a}$$

$$G_p(\xi_s, 0) = 0 \tag{13b}$$

$$\xi(\xi_s, 0) = \xi_s \tag{13c}$$

Formulations similar to that represented by Eqs. (11-13) have been previously employed in the special cases of the thin wedge (Peddieson and Lyu[4]), the sphere (Peddieson and Lyu[5]), and the cone (Lyu and Peddieson[6]). The convenience of this formulation in these cases motivated the present work.

Equations (11) combine the main feature of the Lagrangian description used by Probstein and Fassio,[1] Waldman and Reinecke,[2] and Spurk and Gerber[3] (i.e., that the flow variables can be determined from the solutions of ordinary differential equations) with the main feature of the Eulerian description (i.e., that the shock surface and the body surface correspond to known values of the independent variable). In addition, the distance from the shock to the body is constant when the variable z is used, and Eq. (11e), which determines the density, is algebraic. For these reasons it is believed that Eqs. (11) provide certain advantages over previous formulations where numerical solutions are to be obtained, and it is hoped that it will be found useful for this purpose by future investigators. The advantage over solution of the original Eulerian system [Eqs. (2)] lies in the fact that three ordinary differential equations and one algebraic equation must be solved rather than four partial differential equations. Numerical solution of the former set will be much easier and faster than numerical solution of the latter set. The advantage over solution of the Lagrangian equations stems from the fact that in the present formulation the body surface has known coordinates while in the Lagrangian formulation it does not. Since the body surface coordinates are known, solution variables on the body surface can be determined without interpolation and optimum step sizes are easier to choose.

Numerical Solutions

Equations (11a-c) can be solved simultaneously subject to Eqs. (13a, c d) by any numerical method for initial value problems. In the present work a fourth-order Runge-Kutta routine was used. The same method was used to solve Eq. (11d) subject to Eq. (13b). Equation (11e) was solved by approximating the partial derivative appearing therein by a two-point backward difference quotient. The general axisymmetric body shape considered was

$$\eta_b = BR(\xi) \tag{14a}$$

$$R(0) = 0 \tag{14b}$$

$$R(1) = 1 \tag{14c}$$

An approximate description of the gas flowfield was obtained by employing expressions for F, G, and H similar to those suggested by Waldman and Reinecke[2] and by assuming that the shock surface could be represented by the equation $\eta_s = SR(\xi)$ where R is the same function appearing in Eq. (14) and the quantity (S/B) is the same as that for a cone with base-radius-to-length ratio B. (A more accurate description of the gas flowfield could be used without affecting the solution procedure in any way, but this would necessitate numerical analysis of the gas-phase governing equations which the author did not want to undertake in connection with the present work.) Numerical solutions were obtained for the specific body shapes

$$R = \xi^\beta \quad \text{(power law)} \tag{15}$$

$$R = (2\xi - C\xi^2)/(2-C), \quad \text{(parabolic series)} \tag{16}$$

$$R = [(\cos^{-1}(1-2\xi))/\pi]^{1/2}, \quad \text{(Von Karman)} \tag{17}$$

$$R = \{[(1+B^2)^2 - \{2B(1-\xi)\}^2]^{1/2} - (1-B^2)\}/(2B^2), \quad \text{(ogive)} \tag{18}$$

where β and C are constants.

Numerical results are presented in Fig. 2 for the local collection efficiency (the rate at which particulate mass is actually collected by the body surface divided by the rate that would exist in the absence of interphase momentum transfer)

$$E(\xi_b) = \frac{\int_0^{\xi_b} Q_{pb}[F_{pb}\tan(\theta_b) - G_{pb}]\, n_b^j \, d\xi_b}{\int_0^{\xi_b} \tan(\theta_b) n_b^j \, d\xi_b} \tag{19}$$

for bodies with a base-diameter-to-length ratio of one third. For all calculations presented, the perfect-gas specific heat ratio $\gamma = 1.4$, the freestream Mach number $M_\infty = 10$, and the momentum transfer coefficient $\alpha_1 = 1$. It can be seen that E is a decreasing function of ξ_b with the detailed shapes of the curves depending on the particular body to which the results correspond.

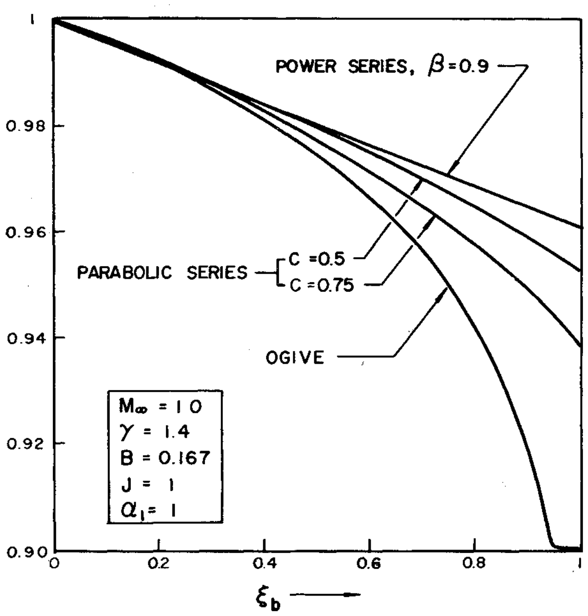

Fig. 2 Collection efficiencies for various body shapes.

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